

# Remarks on the stability of the Navier-Stokes equations supplemented with stress-free boundary conditions

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## Abstract

The purpose of this note is to analyze the long term stability of the Navier-Stokes equations supplemented with the Coriolis force and the stress-free boundary condition. It is shown that, if the flow domain is axisymmetric, spurious stability behaviors can occur depending whether the Coriolis force is active or not.

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## 1. Introduction

The liquid core of the Earth is often modeled as a heated conducting fluid enclosed between the solid inner core and the mantle. Numerically simulating the dynamics of the liquid core is difficult in many respects; one of the difficulties comes from the presence of viscous layers that develop at the boundaries of the fluid domain, i.e., the so-called inner core boundary (ICB) and core mantle boundary (CMB). It is a common practice in the geophysics literature to use stress-free boundary conditions in order to minimize the role played by the viscous layers. Although this choice of boundary condition is convenient, it is not clear that it is more physically justified than using the no-slip condition. Actually, enforcing either the no-slip or the stress-free boundary condition may lead to significantly different results when it comes

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to simulating the geodynamo. For example, Glatzmaier and Roberts [5] and Kuang and Bloxham [8, 9] have used the above two different sets of boundary conditions and have reported numerical buoyancy-driven dynamos in rapidly rotating spherical shells that differ in some fundamental aspects, see e.g. [14]. The simulations reported in [8] use the stress-free condition whereas those reported in [5] use the no-slip condition. The dynamo simulated in [8] is composed of an external magnetic field dominated by an axial dipole component, like that of the Earth, with an intensity close to the present geomagnetic dipole moment. The external magnetic field is comparable to that obtained by Glatzmaier and Roberts [5], but important differences in the velocity and magnetic fields between these two dynamos can be observed within the outer core and the Taylor-Proudman tangent cylinder. (It is known that rotation of the Earth rigidifies the flow field in the direction parallel to the rotation axis through a mechanism known as the Taylor-Proudman effect. This effect makes the imaginary cylinder that is tangent to the equator of the solid inner core and whose axis is parallel to the rotation axis of the Earth act like a solid boundary.) In the dynamo reported in [8] the fluid flow is almost stagnant inside the tangent cylinder and has a strong azimuthal component outside; the magnetic field is composed of two opposite toroidal cells and a simple dipolar poloidal structure and is active throughout the outer core. In the dynamo reported in [5] the fluid flow is composed of an intense polar vortex that is located inside the tangent cylinder and extends in the two hemispheres; the toroidal component of the magnetic field is active only inside the tangent cylinder and is concentrated near the ICB; the poloidal component has a complicated dipolar structure with extra-closed loops near the ICB. It is suggested in [14] that the significantly different structures of the above two dynamos should be attributed to the nature of the boundary conditions that are imposed at the ICB and CMB interfaces.

In addition to thermal or compositional convection due to buoyancy, precession is also believed to be a possible source of energy for the geodynamo. The precession hypothesis has been formulated for the first time in [1] and experimentally investigated using a water model in [11]. It has since then been actively investigated from the theoretical, experimental and numerical perspectives. However, it seems that it is only recently that numerical examples of precession dynamos have been reported in spheres [15, 16], in spheroidal cavities [17] and in cylinders [13]. Recently, Wu and Roberts [17] have numerically studied the dynamo effect in a precessing oblate spheroid. To facilitate their analysis the authors have split the total velocity field into a

basic stationary analytic (polynomial) solution (the so-called Poincaré flow) and a fluctuating part. Following ideas of Kerswell and Mason [12], they have implemented the stress-free boundary condition on the fluctuating component of the velocity in order to reduce the impact of the viscous layers at the rigid boundaries.

The purpose of the present paper is to show that the use of the stress-free boundary condition poses mathematical difficulties. We prove for instance that, if the fluid domain is not axisymmetric, the flow always returns to rest for large times when the stress-free boundary condition is enforced, but this may not be the case when the flow domain is axisymmetric. Various scenarios can occur depending whether the domain undergoes precession or not.

The note is organized as follows. We analyze the stress-free boundary condition in general fluid domains in §2. We show that this boundary condition is admissible if and only if the domain is not axisymmetric (see Proposition 2.2). We revisit the same question in axisymmetric domains that undergo precession in §3 and §4. We show in §3 that the problem exhibits a spurious stability behavior if the stress-free condition is enforced on the velocity field minus the Poincaré flow (i.e., on the perturbation to the Poincaré flow). We show in §4 that the problem always returns to rest for large times if the homogeneous stress-free boundary condition is enforced. The theoretical argumentation developed in §3 and §4 is numerically illustrated in §5. Concluding remarks are reported in §6.

## 2. Stress-free boundary condition without precession

The objective of this section is to investigate the long term stability of the Navier-Stokes equations equipped with the stress-free boundary condition. The fluid domain is denoted  $\Omega$  and is assumed to be open, bounded and Lipschitz.

### 2.1. Position of the problem

We are interested in the motion of an incompressible fluid in a container  $\Omega$  with boundary  $\Gamma$ . The container is assumed to be at rest in a Galilean frame of reference. Denoting  $\mathbf{u}$  the velocity of the fluid and  $p$  the pressure, the fluid motion is modeled by means of the incompressible Navier-Stokes

equations:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) + \nabla p = 0, \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.2)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad (2.3)$$

where  $\nu$  is the kinematic viscosity,  $\boldsymbol{\epsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$  is the strain rate tensor, and  $\mathbf{u}_0$  is an initial data in  $\mathbf{H} := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}_\Gamma = 0\}$ . It is a common practice to replace the term  $\nabla \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T)$  in the momentum equation by  $\Delta \mathbf{u}$  since  $\nabla \cdot \nabla \mathbf{u}^T = 0$  for incompressible flows. We nevertheless keep the original form of the viscous stress since we want to enforce the so-called stress-free boundary condition:

$$(\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u})) \times \mathbf{n}|_\Gamma = 0, \quad (2.4)$$

together with the slip boundary condition:

$$\mathbf{n} \cdot \mathbf{u}|_\Gamma = 0, \quad (2.5)$$

where  $\mathbf{n}$  is the unit outward normal on  $\Gamma$ . The stress-free condition means that the tangent component of the stress at the boundary is zero. We shall see that this boundary condition is admissible in general for non-axisymmetric domains, but it yields pathological stability behaviors if the fluid domain is a solid of revolution.

We are not going to discuss the well-posedness of the above problem in its full generality since it is still unknown whether the three-dimensional Navier-Stokes equations are well-posed under the much simpler no-slip boundary condition. We nevertheless recognize as a symptom of pathological stability behavior the fact that there are solutions to (2.1)-(2.2)-(2.3)-(2.4)-(2.5) that do not return to rest as  $t \rightarrow +\infty$  if  $\Omega$  is axisymmetric.

**Definition 2.1.** *We say that  $\Omega$  is stress-free admissible if there is a constant  $K > 0$ , possibly depending on  $\Omega$ , so that the following holds*

$$K \int_\Omega \mathbf{v}^2 \leq \int_\Omega \boldsymbol{\epsilon}(\mathbf{v}) : \boldsymbol{\epsilon}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v} \cdot \mathbf{n}|_\Gamma = 0 \quad (2.6)$$

where “:” denotes the tensor double product.

**Proposition 2.1.** *Assume that  $\Omega$  is stress-free admissible, then  $\{0\}$  is the global attractor of (2.1)-(2.2)-(2.3)-(2.4)-(2.5).*

*Proof.* We omit the details concerning the existence of Leray-Hopf solutions, which can be constructed using standard Galerkin techniques [10], and we focus only on the aspects of the question which are relevant to our discussion. It is clear that 0 is an invariant set of (2.1)-(2.2)-(2.3)-(2.4)-(2.5). Let  $\mathbf{B}$  be a bounded set in  $\mathbf{H}$  and let  $\mathbf{u}_0 \in \mathbf{B}$ . Let  $\mathbf{u}$  be a Leray-Hopf solution corresponding to the initial data  $\mathbf{u}_0$  and let  $\mathbf{v}$  be a smooth solenoidal vector field satisfying the slip boundary condition. Upon multiplying the momentum equation by  $\mathbf{v}$  and integrating over the domain we obtain

$$\int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} + \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} - 2\nu \int_{\Omega} \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{v} + \int_{\Omega} \nabla p \cdot \mathbf{v} = 0.$$

Solenoidality and the slip boundary condition imply that  $\int_{\Omega} \nabla p \cdot \mathbf{v} = - \int_{\Omega} p \nabla \cdot \mathbf{v} + \int_{\Gamma} p \mathbf{v} \cdot \mathbf{n} = 0$ . Now, using the decomposition

$$\mathbf{v} = (\mathbf{n} \cdot \mathbf{v}) \mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{v}),$$

and integrating by parts the viscous term we obtain:

$$\begin{aligned} - \int_{\Omega} \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{v} &= \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) : \nabla \mathbf{v} - \int_{\Gamma} \mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{v} \\ &= \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) + \int_{\Gamma} (\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u}) \times \mathbf{n}) \cdot (\mathbf{n} \times \mathbf{v}) = \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}). \end{aligned}$$

The transport term and the time derivative are re-written in the following form

$$\begin{aligned} \int_{\Omega} \mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} &= \int_{\Omega} \frac{1}{2} \nabla \cdot (\mathbf{u}(\mathbf{u} \cdot \mathbf{v})) + \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u}) = \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{u}), \\ \int_{\Omega} \partial_t \mathbf{u} \cdot \mathbf{v} &= \frac{1}{2} \int_{\Omega} \partial_t (\mathbf{u} \cdot \mathbf{v}) + \frac{1}{2} \int_{\Omega} (\partial_t \mathbf{u} \cdot \mathbf{v} - \partial_t \mathbf{v} \cdot \mathbf{u}). \end{aligned}$$

We now apply the above identities by replacing  $\mathbf{v}$  by a sequence  $\{\mathbf{v}_n\}_{n \in \mathbb{N}}$  that converges in the appropriate norm to  $\mathbf{u}$ . By passing to the limit (we omit the details again), we finally obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbf{u}^2 + 2\nu \int_{\Omega} \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) \leq 0.$$

Note that equality is lost in the passage to the limit. Whether equality holds in general is an open problem which is part of the Millenium prize. Then

using (2.6), we infer the following inequality:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \mathbf{u}^2 + 2K\nu \int_{\Omega} \mathbf{u}^2 \leq 0,$$

which immediately leads to

$$\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} e^{-2K\nu t},$$

thereby proving that  $\mathbf{u} \rightarrow 0$  as  $t \rightarrow +\infty$ .  $\square$

We shall see that the stress-free admissibility condition (2.6) does not hold for axisymmetric fluid domains, which are common in geoscience.

## 2.2. The non-axisymmetric case

To better understand the stress-free admissibility condition (2.6), we first prove that it holds if and only if  $\Omega$  is not axisymmetric.

**Definition 2.2.** *We say that  $\Omega$  is axisymmetric (or is solid of revolution) if and only if there is a rotation  $\mathbf{R} : \Omega \rightarrow \Omega$  which is tangent on  $\Gamma$ .*

Upon introducing the average operator over  $\Omega$ ,  $\langle v \rangle := \frac{1}{|\Omega|} \int_{\Omega} v$ , where  $|\Omega|$  is the volume of  $\Omega$ , the following lemma gives a characterization of non-axisymmetric domains:

**Lemma 2.1** (Desvillettes-Villani [3]). *Assume that the domain  $\Omega$  is not a solid of revolution of class  $\mathcal{C}^1$ , then there is  $c > 0$  so that*

$$c|\Omega| \langle \nabla \times \mathbf{v} \rangle^2 \leq \|\boldsymbol{\epsilon}(\mathbf{v})\|_{\mathbf{L}^2(\Omega)}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0.$$

We are now in measure to state the main result of this section:

**Proposition 2.2.** *Assume that the domain  $\Omega$  is of class  $\mathcal{C}^1$ , then  $\Omega$  is stress-free admissible if and only if  $\Omega$  is not a solid of revolution.*

*Proof.* Let us assume first that  $\Omega$  is not a solid of revolution and (2.6) does not hold. We start from the Korn inequality (cf. e.g. [4]): there exists a constant  $c > 0$  such that, for all  $\mathbf{v} \in \mathbf{H}^1(\Omega)$ ,

$$\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq c \left( \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{v} + \nabla \mathbf{v}^T\|_{\mathbf{L}^2(\Omega)} \right). \quad (2.7)$$

Since (2.6) does not hold, for any  $n \in \mathbb{N}$ , one can find  $\mathbf{u}_n \in \mathbf{H}^1(\Omega)$  such that

$$\mathbf{u}_n \cdot \mathbf{n}|_{\Gamma} = 0, \quad \|\mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} = 1, \quad \text{and} \quad \|\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^T\|_{\mathbf{L}^2(\Omega)} \leq \frac{1}{n}.$$

The Korn inequality implies that the sequence  $\mathbf{u}_n$  is bounded in  $\mathbf{H}^1(\Omega)$ . Since the inclusion  $\mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega)$  is compact, there exists  $\mathbf{u} \in \mathbf{H}^1(\Omega)$  such that (we keep using  $\mathbf{u}_n$  after extraction of the converging sub-sequence)  $\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$  and  $\mathbf{u}_n \rightharpoonup \mathbf{u}$  in  $\mathbf{H}^1(\Omega)$ . We also have

$$\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^T \rightarrow 0 \text{ in } \mathbf{L}^2(\Omega) \text{ and } \nabla \mathbf{u}_n + \nabla \mathbf{u}_n^T \rightarrow \nabla \mathbf{u} + \nabla \mathbf{u}^T \text{ in } \mathcal{D}'(\Omega),$$

which finally gives  $\nabla \mathbf{u} + \nabla \mathbf{u}^T = 0$  ( $\mathcal{D}(\Omega)$  is the space of smooth vector-valued functions with compact support in  $\Omega$  and  $\mathcal{D}'(\Omega)$  is the space of vector-valued distributions over  $\Omega$ , i.e., the linear forms acting on  $\mathcal{D}(\Omega)$ .) Applying the Korn inequality to  $\mathbf{u} - \mathbf{u}_n$  and using the fact that

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{u}_n + \nabla \mathbf{u}_n^T - \nabla \mathbf{u} - \nabla \mathbf{u}^T\|_{\mathbf{L}^2(\Omega)} \rightarrow 0,$$

we infer that  $\|\mathbf{u}_n - \mathbf{u}\|_{\mathbf{H}^1(\Omega)} \rightarrow 0$ . This allows us to pass to the limit on the boundary condition  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0$ . The condition  $\epsilon(\mathbf{u}) = 0$  implies that there are two vectors  $\mathbf{t} \in \mathbb{R}^3$ ,  $\boldsymbol{\omega} \in \mathbb{R}^3$  so that  $\mathbf{u} = \mathbf{t} + \boldsymbol{\omega} \times \mathbf{x}$ . This means that  $\nabla \times \mathbf{u} = \langle \nabla \times \mathbf{u} \rangle = \boldsymbol{\omega}$ . Using Lemma (2.1), we conclude that  $\boldsymbol{\omega} = 0$ , which means that  $\mathbf{u} = \mathbf{t}$ . The boundary condition  $\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0$  implies  $\mathbf{t} = 0$ ; this in turn means  $\mathbf{u} = 0$ , which is impossible because  $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} = 1$ . In conclusion, (2.6) holds.

Let us assume now that  $\Omega$  is axisymmetric. This means that there is a rotation  $\mathbf{R} : \Omega \rightarrow \Omega$  which is tangent on  $\Gamma$ . Let us assume that the rotation axis is parallel to  $\mathbf{e}_z$  and the coordinate origin is located on this axis. Then  $\mathbf{R}(\mathbf{x}) = \omega \mathbf{e}_z \times \mathbf{x}$  and clearly  $\mathbf{R} \in \mathbf{H}^1(\Omega)$ ,  $\mathbf{R}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})|_{\Gamma} = 0$ ,  $\|\mathbf{R}\|_{\mathbf{L}^2(\Omega)} \neq 0$  but (2.6) does not hold since  $\epsilon(\mathbf{R}) = 0$ .  $\square$

### 2.3. The Axisymmetry curse

Let us assume that  $\Omega$  is axisymmetric. We are going to show the following statement in this section.

**Claim 2.1.** *The zero velocity field, 0, is in the global attractor of (2.1)-(2.2)-(2.3)-(2.4)-(2.5), but the rest state,  $\{0\}$ , is not an attractor. There are initial data that create flows that never return to rest. In particular, if the initial data is a solid rotation, the flow will rotate for ever without losing energy.*

Recall that it can be shown that  $\Omega$  is axisymmetric if and only if  $\Omega$  is either a sphere (and all the directions are symmetry axes) or  $\Omega$  has a unique symmetry axis. Without a loss of generality, we assume  $Oz$  is the only symmetry axis of  $\Omega$ . Recall that all the solid rotations about  $Oz$  can be written as follows  $\mathbf{x} \mapsto \omega \mathbf{e}_z \times \mathbf{x}$ ,  $\omega \in \mathbb{R}$ , where  $\mathbf{x}$  is the position vector. We introduce the following space

$$\mathcal{R} := \text{span} \{ \mathbf{e}_z \times \mathbf{x} \} \quad (2.8)$$

and its orthogonal in  $\mathbf{L}^2(\Omega)$ , say  $\mathcal{R}^\perp$ .

**Lemma 2.2.** *Let  $\Omega$  be an open, bounded, connected, domain of class  $\mathcal{C}^1$  with unique symmetry axis  $Oz$ . There exists  $K > 0$  such that, for every  $\mathbf{v} \in \mathcal{R}^\perp \cap \mathbf{H}^1(\Omega)$  with  $\mathbf{v} \cdot \mathbf{n} = 0$*

$$K \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 \leq \int_{\Omega} |\boldsymbol{\epsilon}(\mathbf{v})|^2,$$

where we denote  $|\boldsymbol{\epsilon}(\mathbf{v})|^2 := \boldsymbol{\epsilon}(\mathbf{v}) : \boldsymbol{\epsilon}(\mathbf{v})$ .

*Proof.* The proof is similar to that of Proposition 2.2. By contradiction, we consider a sequence  $\mathbf{v}_n \in \mathcal{R}^\perp \cap \mathbf{H}^1(\Omega)$  with vanishing normal component such that

$$\|\mathbf{v}_n\|_{\mathbf{L}^2} = 1 \text{ and } \|\boldsymbol{\epsilon}(\mathbf{v}_n)^T\|_{\mathbf{L}^2} \leq \frac{1}{n}.$$

Using Korn inequality, we can prove that (up to extraction)  $\mathbf{v}_n$  converges in  $\mathbf{H}^1(\Omega)$ , and the limit  $\mathbf{v}$  satisfies

$$\mathbf{v} \in \mathcal{R}^\perp, \mathbf{v} \cdot \mathbf{n} = 0 \text{ and } \boldsymbol{\epsilon}(\mathbf{v}) = 0.$$

This implies that  $\mathbf{v}$  is the sum of a translation plus a solid rotation. But  $\Omega$  being bounded the translation is zero and  $\mathbf{v}$  is a solid rotation about  $Oz$ -axis (recall that  $\beta \neq 0$ ), i.e.,  $\mathbf{v} \in \mathcal{R} \cap \mathcal{R}^\perp = \{0\}$ , which contradicts  $\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} = 1$ .  $\square$

We claim that the Navier-Stokes problem (2.1)-(2.2) equipped with boundary conditions (2.4)-(2.5) has spurious stability properties due to the following proposition.

**Proposition 2.3.** *(i)  $\mathcal{R}$  is the global attractor of (2.1)-(2.2)-(2.3)-(2.4)-(2.5). (ii) No element in  $\mathcal{R}$  is an attractor.*



*Proof.* (i) Let  $\mathbf{u} \in L^2((0, +\infty); \mathbf{L}^2(\Omega)) \cap L^\infty((0, +\infty); \mathbf{H}^1(\Omega))$  be a Leray-Hopf solution of (2.1)–(2.5) and consider the following decomposition:

$$\mathbf{u}(t) = \mathbf{u}^\perp(t) + \lambda(t)\mathbf{e}_z \times \mathbf{x}, \text{ where } \mathbf{u}^\perp(t) \in \mathcal{R}^\perp, \lambda(t) \in \mathbb{R}, \forall t \in [0, +\infty).$$

$\mathbf{u}$  being a Leray-Hopf solution implies that

$$\|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)}^2 + \lambda(t)^2 \|\mathbf{e}_z \times \mathbf{x}\|_{\mathbf{L}^2(\Omega)}^2 + 4\nu \int_0^t \int_\Omega |\boldsymbol{\epsilon}(\mathbf{u})|^2 \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2.$$

which owing to Lemma 2.2 implies

$$\|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)}^2 + \lambda(t)^2 \|\mathbf{e}_z \times \mathbf{x}\|_{\mathbf{L}^2(\Omega)}^2 + 4\nu K \int_0^t \|\mathbf{u}^\perp\|_{\mathbf{L}^2(\Omega)}^2 d\tau \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2.$$

Using the Gronwall-Bellmann inequality, we infer that  $\|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} e^{-2\nu K t}$ . Invoking Lemma 3.1 we infer that  $\frac{d\lambda(t)}{dt} = 0$ , implying that  $\lambda(t) = \lambda(0)$ . In conclusion

$$\|\mathbf{u}(t) - \lambda_0 \mathbf{e}_z \times \mathbf{x}\|_{\mathbf{L}^2(\Omega)} = \|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} e^{-2\nu K t}.$$

This implies that the global attractor, say  $\mathcal{A}$ , is such that  $\mathcal{A} \subset \mathcal{R}$ , but since  $\lambda_0$  spans  $\mathbb{R}$ , we conclude that  $\mathcal{A} = \mathcal{R}$ .

(ii) Let us consider the solid rotation field  $\mathbf{u} = \omega \mathbf{e}_z \times \mathbf{x} \in \mathcal{R}$ . It is clear that  $\mathbf{u}$  is invariant, i.e., is a steady-state solution. Let  $\mathbf{B}(\mathbf{u}, \rho) \in \mathbf{H}$  be the ball centered at  $\mathbf{u}$  of arbitrary radius  $\rho > 0$ . Let  $\mathbf{v} = \mu \mathbf{e}_z \times \mathbf{x} \in \mathcal{R}$ ,  $\mu \neq 0$ , be another solid rotation and assume that  $\mu$  is small enough so that  $\mathbf{u} + \mathbf{v} \in \mathbf{B}(\mathbf{u}, \rho)$ . Let us observe that

$$(\mathbf{u} + \mathbf{v}) \cdot \nabla(\mathbf{u} + \mathbf{v}) = 2(\mathbf{u} + \mathbf{v}) \cdot \boldsymbol{\epsilon}(\mathbf{u} + \mathbf{v}) - (\mathbf{u} + \mathbf{v}) \cdot (\nabla(\mathbf{u} + \mathbf{v}))^T = -\frac{1}{2} \nabla |\mathbf{u} + \mathbf{v}|^2,$$

since  $\mathbf{u} + \mathbf{v}$  is a solid rotation; moreover,  $\mathbf{u} + \mathbf{v}$  satisfies (2.2), (2.5) and  $\boldsymbol{\epsilon}(\mathbf{u} + \mathbf{v}) = 0$ . The property  $\boldsymbol{\epsilon}(\mathbf{u}) = 0$  implies (2.4) and  $\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u})) = 0$ . Upon setting  $p = \frac{1}{2} |\mathbf{u} + \mathbf{v}|^2$  we conclude that  $\mathbf{u} + \mathbf{v}$  solves (2.1). This proves that  $\mathbf{u} + \mathbf{v}$  is invariant (i.e., a steady-state solution). In other words  $\mathbf{u} + \mathbf{v}$  does not converge to  $\mathbf{u}$ , no matter how small  $\rho$  is, thereby proving that the set  $\{\mathbf{u}\}$  is not an attractor, no matter how large  $\nu$  is.  $\square$

#### 2.4. An admissible stress-free-like boundary condition

The principal motivation to consider the so-called stress-free boundary condition is that it minimizes viscous layers and is thus less computationally demanding than the no-slip boundary condition. We have seen above that this boundary condition unfortunately leads to pathological stability properties when the computational domain is axisymmetric. A possible remedy to this problem is to consider the following non-symmetric boundary condition:

$$(\mathbf{n} \cdot \nabla \mathbf{u}) \times \mathbf{n}|_{\Gamma} = 0. \quad (2.9)$$

The tangent components of the normal derivative of the velocity field are zero. The physical interpretation of this condition is definitely less appealing than that of the stress-free boundary condition, but once one realizes that the stress-free boundary condition is ad hoc, one comes to think that (2.9) is not more ad hoc than the stress-free condition. The main advantage we see in (2.9) over the stress-free condition is that it yields standard stability properties, i.e.,  $\{0\}$  is the global attractor when there is no forcing.

**Lemma 2.3.** *The following holds for all smooth solenoidal vector field  $\mathbf{u}$  that satisfies  $(\mathbf{n} \cdot \nabla \mathbf{u}) \times \mathbf{n}|_{\Gamma} = 0$ :*

$$\int_{\Omega} -\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u})) \cdot \mathbf{v} = \frac{1}{2} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0. \quad (2.10)$$

*Proof.* Upon observing that  $\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u})) = \frac{1}{2} \nabla \cdot (\nabla \mathbf{u})$  since  $\mathbf{u}$  is solenoidal, we infer that

$$\begin{aligned} \int_{\Omega} -\nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u})) \cdot \mathbf{v} &= \int_{\Omega} -\frac{1}{2} \nabla \cdot (\nabla \mathbf{u}) \cdot \mathbf{v} = \frac{1}{2} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \frac{1}{2} \int_{\Gamma} (\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} \\ &= \frac{1}{2} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} - \frac{1}{2} \int_{\Gamma} (\mathbf{n} \cdot \nabla \mathbf{u}) \cdot ((\mathbf{n} \cdot \mathbf{v}) \mathbf{n}) + \frac{1}{2} \int_{\Gamma} ((\mathbf{n} \cdot \nabla \mathbf{u}) \times \mathbf{n}) \cdot (\mathbf{n} \times \mathbf{v}) \\ &= \frac{1}{2} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}, \end{aligned}$$

where we used again the decomposition  $\mathbf{v}|_{\Gamma} = (\mathbf{n} \cdot \mathbf{v}) \mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{v})$ .  $\square$

**Proposition 2.4.** *Assume that  $\Omega$  is an open, connected, bounded Lipschitz domain, then  $\{0\}$  is the global attractor of (2.1)-(2.2)-(2.3)-(2.9)-(2.5).*

*Proof.* Repeat the argument in the proof of Proposition 2.1 using Lemma 2.3 together with the following Poincaré-like inequality

$$K \int_{\Omega} \mathbf{v}^2 \leq \int_{\Omega} |\nabla \mathbf{v}|^2, \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0,$$

which can be shown to hold by proceeding as in the proof of Proposition 2.2.  $\square$

### 3. Precession driven flow with Poincaré stress

If the fluid domain is a spheroid that undergoes precession, the steady state Navier-Stokes equations with the slip condition admit a so-called Poincaré solution. We show in this section that, independently of the value of the viscosity, the Poincaré solution is not an attractor of the problem if the tangential stress at the boundary is enforced to be equal to that of the steady-state Poincaré solution.

#### 3.1. Geometry and equations

The container is an ellipsoid of revolution of center  $O$  and symmetry axis  $Oz$ . The unit vector along the  $Oz$ -axis is  $\mathbf{e}_z$ . The unit vectors along the other two orthogonal axes  $Ox$  and  $Oy$  are  $\mathbf{e}_x$  and  $\mathbf{e}_y$ , respectively. The surface of the ellipsoid is defined by the equation

$$x^2 + y^2 + (1 + \beta)z^2 = 1, \quad (3.1)$$

where  $\beta > -1$  and  $\beta \neq 0$ . We assume that the container rotates about the  $Oz$ -axis with angular velocity  $\mathbf{e}_z$  and that this frame slowly precesses about the  $Ox$ -axis with angular velocity  $\varepsilon \mathbf{e}_x$  (this particular precession angle is investigated in [17]). The non-dimensional Navier-Stokes equations describing the motion of the fluid in the non-inertial precessing frame of reference  $(O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$  are written as follows:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) + 2\varepsilon \mathbf{e}_x \times \mathbf{u} + \nabla p = 0, \quad (3.2)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (3.3)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0. \quad (3.4)$$

We additionally enforce the slip boundary condition,

$$\mathbf{u} \cdot \mathbf{n}|_{\Gamma} = 0. \quad (3.5)$$

The system (3.2)-(3.3)-(3.5) is known to admit a steady solution called the Poincaré flow (see e.g. [17]); its expression is:

$$\mathbf{u}_P = -y\mathbf{e}_x + \left(x - \frac{2\varepsilon}{\beta}(1 + \beta)z\right)\mathbf{e}_y + \frac{2\varepsilon}{\beta}y\mathbf{e}_z. \quad (3.6)$$

Similarly to [17] we consider the problem (3.2)-(3.3)-(3.5) equipped with the additional non-homogeneous boundary condition

$$(\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u})) \times \mathbf{n}|_\Gamma = (\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u}_P)) \times \mathbf{n}|_\Gamma. \quad (3.7)$$

That is, we want the tangential component of the normal stress to be equal to that of the Poincaré solution. As mentioned in [17], it is clear that

**Claim 3.1** (See [17]).  $\mathbf{u}_P$  is a steady state solution of (3.2)-(3.3)-(3.5)-(3.7).

### 3.2. Long term stability

The question that we now want to investigate is whether there is a threshold on  $\nu$  beyond which  $\mathbf{u}_P$  is a stable solution as  $t \rightarrow +\infty$ ; i.e., does the flow return to  $\mathbf{u}_P$  independently of the initial data as  $t \rightarrow +\infty$  if  $\nu$  is large enough? We show in this section that the answer to this question is no, the fundamental reason being that solid rotations cannot be dampened by viscous dissipation, no matter how large  $\nu$  is.

**Proposition 3.1.** *For all  $\nu > 0$ ,  $\{\mathbf{u}_P\}$  is not an attractor of the Navier-Stokes problem (3.2)-(3.3) equipped with the boundary conditions (3.5)-(3.7).*

*Proof.* Let  $\rho > 0$  be an arbitrary positive number. Let  $\mathbf{B}(\mathbf{u}_P, \rho) \subset \mathbf{H}$  be a ball of radius  $\rho$  centered at  $\mathbf{u}_P$ . Let  $\mathbf{w} = \omega \mathbf{e}_z \times \mathbf{r}$  is a solid rotation about the  $Oz$ -axis, and assume that  $\omega \neq 0$  is small enough so that  $\mathbf{u}_P + \mathbf{w} \in \mathbf{B}(\mathbf{u}_P, \rho)$ . Let us prove that  $\mathbf{u}_P + \mathbf{w}$  is a steady state solution of (3.2)-(3.3)-(3.5)-(3.7). Owing to  $\boldsymbol{\epsilon}(\mathbf{w}) = 0$ ,  $\mathbf{w} \cdot \mathbf{n}|_\Gamma = 0$ ,  $\nabla \cdot \mathbf{w} = 0$ , it is clear that  $\mathbf{u}_P + \mathbf{w}$  is solenoidal and satisfies the boundary conditions (3.5)-(3.7). Let us now show that it is possible to find a pressure field so that the steady state momentum equation holds.

Let us first prove that  $\mathbf{u}_P \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_P + 2\varepsilon \mathbf{e}_x \times \mathbf{w}$  is a gradient. A straightforward computation gives:

$$\mathbf{u}_P \cdot \nabla \mathbf{w} = \omega \begin{pmatrix} \frac{2\varepsilon}{\beta}(1 + \beta)z - x \\ -y \\ 0 \end{pmatrix}, \quad \mathbf{w} \cdot \nabla \mathbf{u}_P = \omega \begin{pmatrix} -x \\ -y \\ \frac{2\varepsilon}{\beta}x \end{pmatrix}, \quad 2\varepsilon \mathbf{e}_x \times \mathbf{w} = \omega \begin{pmatrix} 0 \\ 0 \\ 2\varepsilon x \end{pmatrix},$$

so that

$$\mathbf{u}_P \cdot \nabla \mathbf{w} + \mathbf{w} \cdot \nabla \mathbf{u}_P + 2\varepsilon \mathbf{e}_x \times \mathbf{w} = -\nabla (\omega(x^2 + y^2)) + \nabla \left( \frac{2\varepsilon\omega}{\beta}(1 + \beta)xz \right).$$

Let us then define  $q(\mathbf{x}) := -\omega(x^2 + y^2) + \frac{2\varepsilon\omega}{\beta}(1 + \beta)xz$ . Observe that we can define the pressure field  $r(\mathbf{x})$  so that  $\nabla r := -\mathbf{u}_P \cdot \nabla \mathbf{u}_P - 2\varepsilon \mathbf{e}_x \times \mathbf{u}_P$ , since  $\mathbf{u}_P$  solves (3.2). Let us finally observe that  $\mathbf{w} \cdot \nabla \mathbf{w} = -\frac{1}{2} \nabla |\mathbf{w}|^2$ . Then we conclude that  $\mathbf{u}_P + \mathbf{w}$  solves (3.2) with  $p = q + r - \frac{1}{2} |\mathbf{w}|^2$ . In particular if we set  $\mathbf{u}_0 = \mathbf{u}_P + \mathbf{w}$ , then  $\mathbf{u}_P + \mathbf{w}$  remains a solution forever, i.e., the solution does not converge to  $\mathbf{u}_P$  as  $t \rightarrow +\infty$ , no matter how small  $\rho$  is and no matter how large  $\nu$  is.  $\square$

### 3.3. Angular momentum balance

Let us now mention a result on the balance of the angular momentum. Let us assume that  $\mathbf{u}$  solves (3.2)-(3.3) with the boundary conditions

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad \text{on } \Gamma, \quad (3.8)$$

$$(\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u})) \times \mathbf{n} = \mathbf{g} \times \mathbf{n} \quad \text{on } \Gamma, \quad (3.9)$$

where the field  $\mathbf{g}$  is a boundary data. Let us now define the angular momentum

$$\mathbf{M} := \int_{\Omega} \mathbf{x} \times \mathbf{u}. \quad (3.10)$$

**Lemma 3.1.** *Denoting by  $M_z$  and  $M_y$  the  $z$ - and  $y$ -component of  $\mathbf{M}$ , respectively, all the weak solutions of (3.2)-(3.3)-(3.8)-(3.9) satisfy*

$$\partial_t M_z + \varepsilon M_y = - \int_{\partial\Omega} \nu(\mathbf{g} \times \mathbf{n}) \cdot ((\mathbf{e}_z \times \mathbf{x}) \times \mathbf{n}), \quad \text{a.e. } t \in (0, +\infty). \quad (3.11)$$

*Proof.* Observing that  $M_z = \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \mathbf{u}$ , we multiply (3.2) by  $\mathbf{e}_z \times \mathbf{x}$  and integrate over  $\Omega$ . Using the divergence free condition together with (3.8) and integrating by parts, we infer that

$$\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = \int_{\Omega} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) \cdot (\mathbf{e}_z \times \mathbf{x}) = \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) (\mathbf{u} \cdot (\mathbf{e}_z \times \mathbf{x})) = 0,$$

where we used that  $(\mathbf{u} \otimes \mathbf{u}) : \nabla (\mathbf{e}_z \times \mathbf{x}) = 0$  since the matrix  $\mathbf{u} \otimes \mathbf{u}$  is symmetric and  $\nabla (\mathbf{e}_z \times \mathbf{x})$  is anti-symmetric. The same argument applies to the viscous term

$$\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \nu \nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u})) = \int_{\partial\Omega} \nu (\boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{n}) \cdot (\mathbf{e}_z \times \mathbf{x}) = \int_{\partial\Omega} \nu(\mathbf{g} \times \mathbf{n}) \cdot ((\mathbf{e}_z \times \mathbf{x}) \times \mathbf{n}),$$

where we used  $\mathbf{e}_z \times \mathbf{x} = (\mathbf{e}_z \times \mathbf{x}) \times \mathbf{n}$  since  $(\mathbf{e}_z \times \mathbf{x}) \cdot \mathbf{n}|_\Gamma = 0$ . The same argument applies again for the pressure term since  $\nabla p = \nabla \cdot (pI)$  where  $I$  is the identity matrix.

$$\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \nabla p = \int_{\partial\Omega} p (\mathbf{e}_z \times \mathbf{x}) \cdot \mathbf{n} = 0.$$

We now deal with the Coriolis term by applying Lemma 3.2:

$$\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{e}_x \times \mathbf{u}) = \frac{1}{2} \int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times \mathbf{u}) = \frac{1}{2} M_y.$$

The conclusion follows readily.  $\square$

**Lemma 3.2.** *Let  $\mathbf{v} \in \mathbf{L}^1(\Omega)$  be an integrable vector field such that  $\nabla \cdot \mathbf{v} = 0$  and  $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$ , then*

$$\int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times \mathbf{v}) = 2 \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{e}_x \times \mathbf{v}). \quad (3.12)$$

*Proof.* Let us first observe that  $\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{e}_x \times \mathbf{v}) = - \int_{\Omega} x u_z$ . Noticing that  $\int_{\Omega} x v_z + z v_x = \int_{\Omega} \mathbf{v} \cdot \nabla (zx) = 0$  since  $\nabla \cdot \mathbf{v} = 0$  and  $\mathbf{v} \cdot \mathbf{n}|_\Gamma = 0$ , we infer that

$$\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{e}_x \times \mathbf{v}) = - \int_{\Omega} x u_z = \frac{1}{2} \int_{\Omega} z v_x - x v_z = \frac{1}{2} \int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times \mathbf{v}),$$

which concludes the proof.  $\square$

*Remark 3.1.* If we choose  $\mathbf{g} = \boldsymbol{\epsilon}(\mathbf{u}_P) \cdot \mathbf{n}$  like in (3.7), then  $-\int_{\partial\Omega} \nu(\mathbf{g} \times \mathbf{n}) \cdot ((\mathbf{e}_z \times \mathbf{x}) \times \mathbf{n})$  is equal to  $-\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \nu \nabla \cdot (\boldsymbol{\epsilon}(\mathbf{u}_P)) = 0$  and the balance equation of the angular momentum in the  $z$  direction simplifies to  $\partial_t M_z + \varepsilon M_y = 0$ .

*Remark 3.2.* Note that (3.11) is just a consequence of (3.2)-(3.3)-(3.8)-(3.9). This balance holds whether the long term stability of (3.2)-(3.3)-(3.8)-(3.9) is spurious or not. It is false to consider that (3.11) is an additional equation that fixes the long term stability behavior of (3.2)-(3.3)-(3.5)-(3.7).

#### 4. Precession driven flow with stress-free boundary conditions

We show in this section that if we enforce  $\boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{n}|_\Gamma = 0$ , instead of enforcing  $\boldsymbol{\epsilon}(\mathbf{u}) \cdot \mathbf{n}|_\Gamma = \boldsymbol{\epsilon}(\mathbf{u}_P) \cdot \mathbf{n}|_\Gamma$  in (3.2)-(3.3)-(3.8), then 0 becomes the unique stable solution as  $t \rightarrow +\infty$ , i.e.,  $\{0\}$  is the global attractor.

#### 4.1. Long time stability

The setting of the problem is the same as in Section 3.1 except that we enforce the tangential component of the normal stress to be zero at the boundary.

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - 2\nu \nabla \cdot \boldsymbol{\epsilon}(\mathbf{u}) + 2\varepsilon \mathbf{e}_x \times \mathbf{u} + \nabla p = 0 \quad \text{in } \Omega \quad (4.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \quad (4.2)$$

$$\mathbf{n} \cdot \mathbf{u} = 0 \quad \text{on } \Gamma \quad (4.3)$$

$$(\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u})) \times \mathbf{n} = 0 \quad \text{on } \Gamma \quad (4.4)$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega. \quad (4.5)$$

The result that we want to emphasize is that contrary to what we observed in Section 3, 0 becomes the unique stable solution of (4.1)–(4.5) as  $t \rightarrow +\infty$ . The main result that we want to prove here is that any solution of the system (4.1)–(4.4) returns to rest as  $t \rightarrow +\infty$ . The key argument is that solid rotations about the  $Oz$  axis are not stationary solutions of (4.1). This fact has been mentioned in [17] without proof.

**Theorem 4.1.**  $\{0\}$  is the global attractor of (4.1)–(4.5).

*Proof.* Let us start by observing that  $\{0\}$  is indeed an invariant set of (4.1)–(4.5). Let  $\mathbf{B}(0, \rho)$  be the unit ball in  $\mathbf{H}$  centered at 0 and of radius  $\rho$ . Let  $\mathbf{u}_0 \in \mathbf{B}(0, \rho)$  and let  $\mathbf{u} \in L^2((0, +\infty); \mathbf{L}^2(\Omega)) \cap L^\infty((0, +\infty); \mathbf{H}^1(\Omega))$  be a Leray-Hopf solution of (4.1)–(4.5) and consider the following decomposition:

$$\mathbf{u}(t) = \mathbf{u}^\perp(t) + \lambda(t) \mathbf{e}_z \times \mathbf{x}, \text{ where } \mathbf{u}^\perp(t) \in \mathcal{R}^\perp, \lambda(t) \in \mathbb{R}, \forall t \in [0, +\infty).$$

Lemma 2.2 together with  $\mathbf{u}$  being a Leray-Hopf solution implies that

$$\|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)}^2 + \gamma \lambda(t)^2 + 4\nu K \int_0^t \|\mathbf{u}^\perp(\tau)\|_{\mathbf{L}^2(\Omega)}^2 d\tau \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2.$$

where  $\gamma = \|\mathbf{e}_z \times \mathbf{x}\|_{\mathbf{L}^2}^2$ . Using the Gronwall-Bellmann inequality, we infer that  $\|\mathbf{u}^\perp(t)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} e^{-2\nu K t}$ .

Let  $t_2 > t_1$  in  $(0, +\infty)$ , then (3.11) means that

$$(\lambda(t_2) - \lambda(t_1))\gamma = -\varepsilon \int_{t_1}^{t_2} \int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times (\lambda(\tau)(\mathbf{e}_z \times \mathbf{x}) + \mathbf{u}^\perp)).$$

But Lemma 3.2 implying that

$$\begin{aligned} \int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times (\lambda(\tau)(\mathbf{e}_z \times \mathbf{x}))) &= \lambda(\tau) \int_{\Omega} \mathbf{e}_y \cdot (\mathbf{x} \times (\mathbf{e}_z \times \mathbf{x})) \\ &= 2\lambda(\tau) \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\mathbf{e}_x \times (\mathbf{e}_z \times \mathbf{x})) = 0, \end{aligned}$$

we finally infer that

$$|\lambda(t_2) - \lambda(t_1)| \leq \gamma^{-1} \varepsilon \int_{t_1}^{t_2} \int_{\Omega} |\mathbf{e}_y \cdot (\mathbf{x} \times \mathbf{u}^\perp)| \leq c(e^{-2\nu K t_1} - e^{-2\nu K t_2}),$$

where  $c$  is a generic constant that depends on  $\Omega$ ,  $\nu$ , and  $\rho$  and may vary at each occurrence from now on. Note in passing that this also proves that  $\lambda(t)$  converges to a real number  $\lambda_\infty$  as  $t \rightarrow +\infty$ , and  $|\lambda_\infty - \lambda(t)| \leq c e^{-2\nu K t}$ .

Let us take  $\boldsymbol{\varphi} \in \mathcal{D}(\Omega)$  independent of time and divergence-free. Since  $\mathbf{u}$  is a Leray-Hopf solution (recall  $t \mapsto \mathbf{u}(t)$  is continuous in the  $\mathbf{L}^2$ -weak topology) we have

$$\begin{aligned} 0 &= \int_{\Omega} (\mathbf{u}(t_2, \mathbf{x}) - \mathbf{u}(t_1, \mathbf{x})) \cdot \boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x} + \int_{t_1}^{t_2} \int_{\Omega} 2\varepsilon \mathbf{u}(\tau, \mathbf{x}) \cdot (\boldsymbol{\varphi}(\mathbf{x}) \times \mathbf{e}_x) \, d\mathbf{x} \, d\tau \\ &\quad - \int_{t_1}^{t_2} \int_{\Omega} 2\nu \mathbf{u}(\tau, \mathbf{x}) \cdot \nabla \cdot \boldsymbol{\epsilon}(\boldsymbol{\varphi}) \, d\mathbf{x} \, d\tau - \int_{t_1}^{t_2} \int_{\Omega} (\mathbf{u}(\mathbf{x}) \otimes \mathbf{u}(\mathbf{x})) : \nabla \boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x} \, d\tau. \end{aligned}$$

Let us now set  $t_2 = t_1 + 1$ . Upon observing that  $\int_{\Omega} ((\mathbf{e}_z \times \mathbf{x}) \otimes (\mathbf{e}_z \times \mathbf{x})) : \nabla \boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x} = 0$  and  $\int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \nabla \cdot \boldsymbol{\epsilon}(\boldsymbol{\varphi}) \, d\mathbf{x} = 0$ . This implies that there is a constant  $c(\boldsymbol{\varphi}) \geq 0$  so that

$$\begin{aligned} 2\varepsilon \left| \int_{t_1}^{t_2} \lambda(\tau) \, d\tau \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\boldsymbol{\varphi}(\mathbf{x}) \times \mathbf{e}_x) \, d\mathbf{x} \right| &\leq \left| (\lambda(t_2) - \lambda(t_1)) \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x}) \, d\mathbf{x} \right| \\ &\quad + c(\boldsymbol{\varphi}) e^{-2\nu K t_1}. \end{aligned}$$

Let us choose  $\boldsymbol{\varphi}$  so that  $2\varepsilon \int_{\Omega} (\mathbf{e}_z \times \mathbf{x}) \cdot (\boldsymbol{\varphi}(\mathbf{x}) \times \mathbf{e}_x) \, d\mathbf{x} = 1$ . The above estimate implies that

$$\left| \int_{t_1}^{t_1+1} \lambda(\tau) \, d\tau \right| \leq c(\boldsymbol{\varphi}) e^{-2\nu K t}.$$

This in turn implies that  $\lambda_\infty = \lim_{t_1 \rightarrow \infty} \int_{t_1}^{t_1+1} \lambda(\tau) \, d\tau = 0$ , which means  $\lambda_\infty = 0$ . In conclusion

$$\lim_{t \rightarrow +\infty} \|\mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} \leq c(\boldsymbol{\varphi}) \lim_{t \rightarrow +\infty} e^{-2\nu K t} = 0, \quad (4.6)$$

which concludes the proof.  $\square$



## 5. Numerical illustrations

To illustrate the above mathematical results, we have performed two series of numerical simulations similar to those presented in [17]. The authors study therein the dynamo action in an oblate spheroid defined by equation (3.1) with  $\beta = 0.5625$  (this corresponds to the value  $b = 0.8$  for the semi-minor axis used in [17],  $b := (1 + \beta)^{-\frac{1}{2}}$ ). This spheroid rotates about the  $Oz$ -axis and precesses about the  $Ox$ -axis. Two sets of boundary conditions are considered: either the homogeneous stress-free boundary or the Poincaré stress condition is enforced. Simulations are carried out using a mixed Fourier decomposition and finite element code described in details in [6].

The first simulation solves the equations (4.1)-(4.2)-(4.3)-(4.4) with the initial data  $\mathbf{u}|_{t=0} = 0.1(-y\mathbf{e}_x + x\mathbf{e}_y)$ . The precession rate is  $\epsilon = 0.25$  and the reciprocal of the viscosity is  $1/\nu = 0.024$ . Figure 1 shows the time derivative of the total energy  $E_K = \frac{1}{2}\|\mathbf{u}\|_{\mathbf{L}^2}^2$  in the precessing frame. Note that  $\partial_t E_K$  is always negative, establishing that  $E_K$  is a decreasing function. This graph is in excellent agreement with figure 1 of [17]. It also shows that  $\mathbf{u} \rightarrow 0$  as  $t \rightarrow \infty$  in agreement with (4.6) (i.e.,  $\{0\}$  is indeed the attractor) .

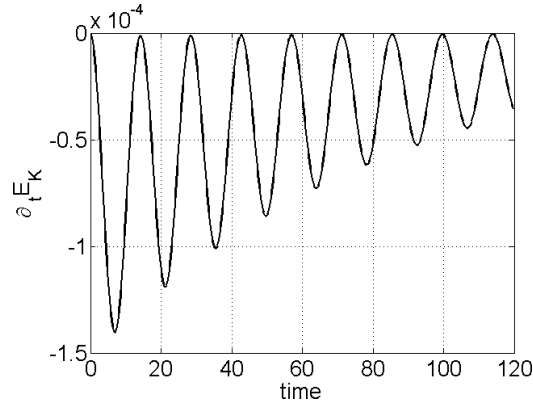


Figure 1: Time evolution of  $\partial_t E_K$  of the solution of equations (4.1)-(4.2)-(4.3)-(4.4) for  $\beta = 0.5625$ ,  $\epsilon = 0.25$  and  $1/\nu = 0.024$ .

The second series of simulations solves equations (3.2)-(3.3)-(3.5)-(3.7) with an *ad hoc* initial condition which is a very small perturbation of the Poincaré solution. The parameters are  $\epsilon = 0.25$  and  $1/\nu = 0.00375$ . Figure 2 shows the time evolution of the kinetic energy of the perturbation to the Poincaré solution,  $\delta E_K = \frac{1}{2}\|\mathbf{u} - \mathbf{u}_P\|_{\mathbf{L}^2(\Omega)}^2$ , from  $t = 0$  to  $t = 1100$ , see curve labeled “0 perturb”. The energy grows exponentially initially, then saturates

around an oscillatory state. These results are similar to those shown in figure 1 of [17].

In order to evaluate the influence of solid rotations, we restart the computation at  $t = 1100$  by adding the perturbation  $\pm 0.025(-y\mathbf{e}_x + x\mathbf{e}_y)$  to the solution. Since the maximum norm of the Poincaré solution is 1.25, the added perturbations are only 2% of the maximum velocity. Time integration is performed in each case until convergence to an oscillating state is obtained. The curves corresponding to the time evolution of the kinetic energy of the solutions thus obtained are labeled “0.025 perturb” and “-0.025 perturb” in Figures 2-2(b). We observe in Figure 2(b) that these perturbations have strong impacts on the asymptotic solutions. To better compare our results

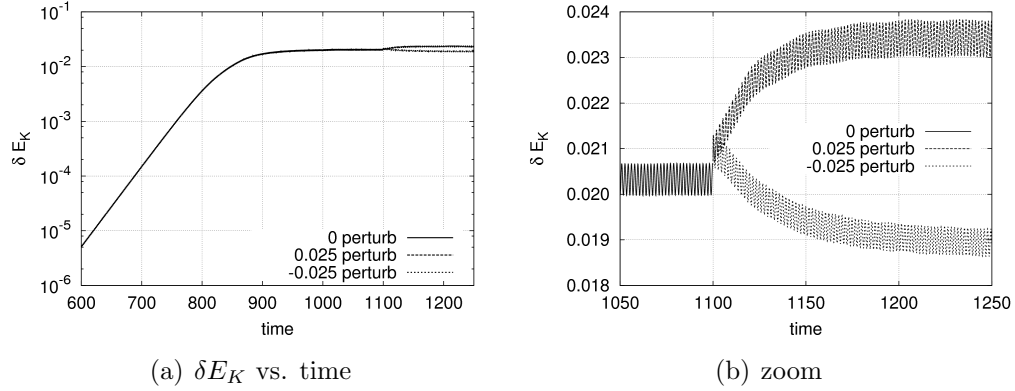


Figure 2: (Color online) Time evolution of the kinetic energy,  $\delta E_K$ , of the perturbation of the solution of (3.2)-(3.3)-(3.5)-(3.7) with  $\beta = 0.5625$ ,  $\epsilon = 0.25$  and  $1/\nu = 0.00375$  (a), and zoom (b).

with those from [17], we show in figure 3 the energies in the northern  $\delta E_{K_n}$  and southern  $\delta E_{K_s}$  hemispheres of the spheroid. The perturbation with the positive sign increases both the total kinetic energy and the amplitude of the northern and southern energies, whereas the perturbation with the negative sign decreases both the total kinetic energy and the amplitude of the oscillations of the northern and southern energies. The oscillations of the northern and southern energies obtained with the positive perturbation are more sinusoidal than those obtained with the negative perturbation. The shape of the oscillations of the northern and southern energies obtained with the negative perturbation are similar to those in [17] (see the more pronounced nonlinear shape). These simulations illustrate well that the 0-perturbation solution is

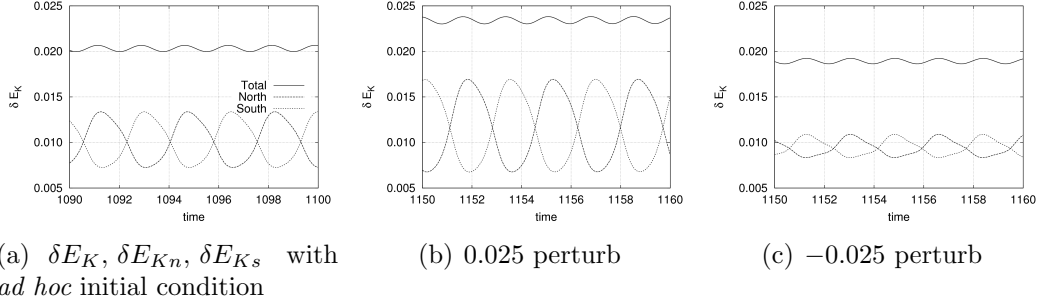


Figure 3: (Color online) Kinetic energy,  $\delta E_K = \frac{1}{2} \|\mathbf{u} - \mathbf{u}_P\|_{\mathbf{L}^2(\Omega)}^2$ , where  $\mathbf{u}$  solves (3.2)-(3.3)-(3.5)-(3.7) with  $\beta = 0.5625$ ,  $\epsilon = 0.25$  and  $1/\nu = 0.00375$ : on each graph, top curve is  $\delta E_K$ , bottom curves are the energies  $\delta E_{K_n}$  (dashed line) and  $\delta E_{K_s}$  (dotted line) in the northern and southern hemispheres.

not an attractor, i.e., it is not stable under perturbations. We have verified (results not shown here) that an entire family of solutions can be obtained from the 0.025-perturbation to the  $-0.025$ -perturbation solutions by scaling the perturbation appropriately. These tests show that using the stress-free boundary condition to evaluate nonlinear behaviors of Navier-Stokes systems may sometimes be dubious when the domain is axisymmetric.

## 6. Discussion

The so-called stress-free boundary condition  $(\mathbf{n} \cdot \boldsymbol{\epsilon}(\mathbf{u})) \times \mathbf{n}|_{\Gamma} = 0$  is often used in the geodynamo literature to avoid issues induced by viscous layers. For example, very recently, an anelastic dynamo benchmark [7] was conducted in a rotating spherical shell. The authors emphasize in their concluding section the difficulties they encountered to compare four different codes using a model with stress-free boundary conditions applied to the ICB and the CMB. Since the container is a spherical shell, the balance equation (3.11) gives  $\partial_t \mathbf{M} = 0$ , and each group had to apply some remedy in order to numerically conserve the three components of the angular momentum. But, more importantly, they also had to use the same initial condition. There was no such difficulties in the older dynamo benchmark [2] using the same geometry because the no-slip boundary condition was prescribed at each interface. These results illustrate again that the stress-free boundary condition induces pathological stability behaviors when the flow domain is axisymmetric.

We have shown in this work that stress-free boundary condition leads to

spurious behaviors when the fluid domain is axisymmetric. We hope that the present work will help draw the attention of the geodynamo community on this problem. The above pathological stability behaviors can be avoided by enforcing one additional condition. For instance, for problem (3.2)–(3.5) and (3.7), one could think of enforcing the vertical component of the angular momentum of the perturbation to the Poincaré flow, say

$$\int_{\Gamma} (\mathbf{u} - \mathbf{u}_P) \cdot (\mathbf{e}_z \times \mathbf{x}) \, ds = 0, \quad (6.1)$$

or enforcing the perturbation and the Poincaré flow to be orthogonal in average over the boundary, say

$$\int_{\Gamma} (\mathbf{u} - \mathbf{u}_P) \cdot \mathbf{u}_P \, ds = 0. \quad (6.2)$$

For problem (2.1)–(2.5), one could think of enforcing the vertical component of the total angular momentum

$$\int_{\Gamma} \mathbf{u} \cdot (\mathbf{e}_z \times \mathbf{x}) \, ds = 0, \quad (6.3)$$

as was done for the three components in the anelastic dynamo benchmark [7].

We have suggested in §2.4 to use a boundary condition that does not have the stability problems mentioned above. For the problem (3.2)–(3.5) this condition is

$$(\mathbf{n} \cdot \nabla \mathbf{u}) \times \mathbf{n}|_{\Gamma} = (\mathbf{n} \cdot \nabla \mathbf{u}_P) \times \mathbf{n}|_{\Gamma}, \quad (6.4)$$

and for the problem (2.1)–(2.5) this condition is

$$(\mathbf{n} \cdot \nabla \mathbf{u}) \times \mathbf{n}|_{\Gamma} = 0. \quad (6.5)$$

Let us finally emphasize that it is false to consider that the momentum balance equation (3.11) is an additional equation that makes (3.2)–(3.3)–(3.5)–(3.7) a well-behaved dynamical system. The equation (3.11) is a redundant consequence of (3.2)–(3.3)–(3.5)–(3.7). For instance, (6.1) (or (6.2) or (6.3)) is an additional equation whereas (3.11) is not.

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